General Relativity

Final Exam
2/11/2012

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1 Problem #1

Consider a 2-dimensional manifold with Euclidean metric $g_{\mu\nu}$.

i) How many independent components does the metric have? How many the Christoffel symbols and how many the Riemann tensor?

ii) Show that in two dimensions the Riemann tensor has the form

$$R_{\mu\nu ab} = \frac{1}{2}R(g_{\mu a}g_{\nu b} - g_{\mu b}g_{\nu a})$$

where R is the Ricci scalar.

iii) Argue that in 2-dimensions we can find coordinates in which the metric locally (in an open set) takes the form

$$ds^2 = A(r,\phi)(dr^2 + r^2 d\phi^2)$$

for some function $A(r,\phi)$.

From this point on, and to simplify the computations, assume that the function $A(r,\phi)$ only depends on r.

iv) Express the Ricci scalar of the metric in terms of $A(r)$.

v) Consider a circle C of constant (coordinate) radius r. Argue that if we start with a vector X^μ at one point on this cirle and we parallel transport it around C, then when we come back to p we get a vector which has the same magnitude as X^μ, but is rotated by an angle ψ.

vi) Parametrize the circle C by some parameter λ such that the tangent vector has constant length along C. By evaluating the covariant derivative of the tangent vector along C, derive a formula for the angle ψ in terms of the Christoffel symbols.

vii) Consider the disk $\mathcal{D}$ which is the interior of the circle C. find the relation between ψ and the quantity

$$\int_{\mathcal{D}} dr d\phi \sqrt{g} R$$

where $\sqrt{g} = \sqrt{det(g_{\mu\nu})}$. Here det is the determinant of the metric,

viii) Consider a two-dimensional sphere of radius r and a circle C corresponding to the equator of the sphere. Verify that the formula you derived in vii) gives the correct result (reminder: for a sphere of radius r we have $R = 2/r^2$).

2 Probelm #2

Consider the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Calculate the escape 4-velocity from this gravitational field. This means: we shoot radially outwards a massive particle from some initial position $r = r_0$, with an initial 4-velocity u^μ. For any given r_0 we want to find the minimum value of u^μ (more precisely, the smallest value of the radial component of u^μ) so that the particle asymptotically reaches infinity with zero spatial velocity. Notice what happens when r_0 approaches the horizon.

3 Problem #3

Consider the metric

$$ds^2 = (1 - r^2)dt - \frac{dr^2}{1 - r^2} - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

which is a solution of Einstein equation with positive cosmological constant.

i) Consider a massive particle undergoing inertial motion in this metric and derive its equations of motion.

From this point on, we will be mostly concerned with motion in the (t, r) coordinates, so in what follows you can assume that the coordinates θ, ϕ always remain fixed.

ii) Show that a particle initially placed at the origin (i.e. $r|_{t=0} = 0$) with zero velocity (i.e. with $\frac{dr}{dt}|_{t=0} = 0$) will always stay there.

iii) Show that particles away from $r = 0$ feel a force towards larger values of r and will thus move towards the surface $r = 1$.

On the surface $r = 1$ the metric has a singularity. This surface is called the *de Sitter horizon*

iv) Write an expression for the the proper distance from $r = 0$ to the de Sitter horizon and show that it is finite.

v) Consider a light ray emitted from $r = 0$ towards the de Sitter horizon. Calculate its orbit and show that in the (t, r) coordinates the light ray never crosses the horizon.

vi) As in the case of the Schwarzschild black hole, this is somewhat misleading. Define the analogue of Eddington Finkelstein coordinates $\bar{t}, r$, in which the outgoing light rays (i.e. those moving towards large values of r) move along straight lines $\bar{t} - r =$ constant. Write the metric (1) in these new coordinates and show that it is smooth at $r = 1$ and can be extended past this surface to $r > 1$.

vii) Show that a freely falling massive particle will cross the de Sitter horizon (the surface $r = 1$) in finite *proper time*, even though we found in v) that the *coordinate time* t necessary for the crossing would be infinite. This situation is very similar to the one we found in the case of the Schwarzschild black hole.

vii) Analyze the causal structure of the metric (1). This means, calculate the form of the in- and out-going lightcones, draw a spacetime diagram in the $\bar{t}, r$ coordinates and plot *qualitatively* the form of the lightcones for ingoing (moving towards smaller r) and outgoing (moving towards larger r) lightrays.

viii) From this diagram, argue that the surface $r = 1$ does indeed act like a horizon, that is, any object which starts in the region $r < 1$ and then crosses the surface $r = 1$ towards $r > 1$, will never be able to come back to the region $r < 1$. From the persepctive of an observer sitting at $r = 0$ this object is forever lost behind the de Sitter horizon.